SUGGESTED SOLUTIONS

Question 1 **(5 marks)**

Consider the polynomial $p(x) = x^3 + ax^2 + bx + 1$, where a and b are **real** numbers.

The polynomial $p(x)$ has **the same** remainder when divided by $(x+2)$ as when divided by $(x-i)$, where $i = \sqrt{-1}$.

Determine the values of a and b.
(Hint: The remainder theorem is also true for complex numbers!)

$p(-2) = -8 + 4a - 2b + 1$

$p(i) = -i - a + bi + 1$

$-7 + 4a - 2b = i(b-1) - a + 1$

$\therefore \boxed{b = 1}$

$-9 + 4a = -a + 1$

$5a = 10$

$\boxed{a = 2}$

Question 2 **(10 marks)**

Let $f(x) = \sqrt{4x - 3}$.

(a) State the domain of $f(x)$. (1 mark)

$4x - 3 \geq 0$

$x \geq \frac{3}{4}$

(b) Find an expression for the inverse function $f^{-1}(x)$. (2 marks)

$y = \sqrt{4x-3}$

$x = \sqrt{4y-3}$

$x^2 = 4y - 3$

$x^2 + 3 = 4y$

$\therefore f^{-1}(x) = \frac{x^2+3}{4}$

(c) Use algebra to determine the coordinates of the points where the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect. (3 marks)

$x = \sqrt{4x-3}$

$x^2 = 4x - 3$

$x^2 - 4x + 3 = 0$

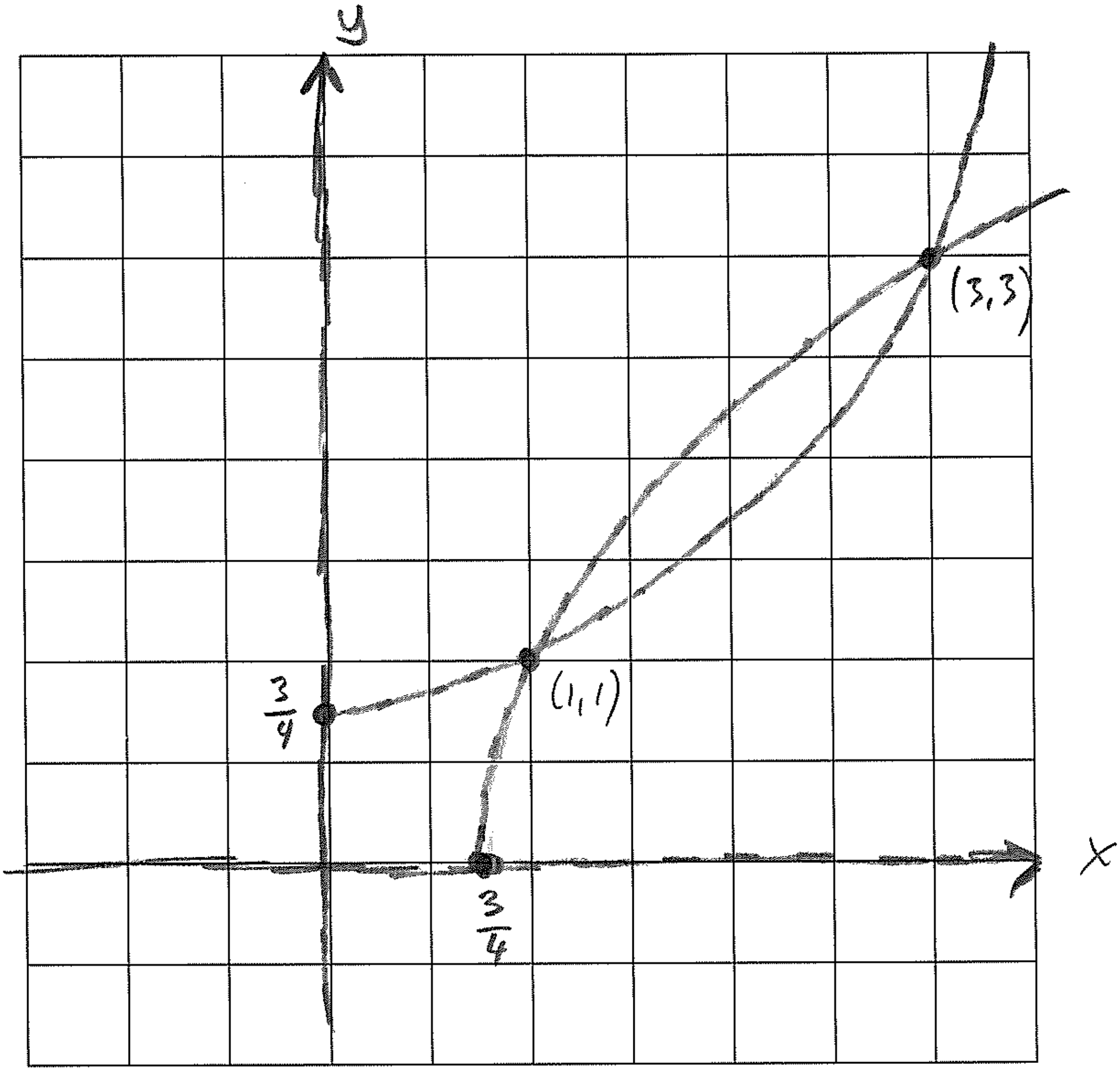
$(x-1)(x-3) = 0$

$\therefore x = 1, 3$

$\therefore$ Intersection points are $(1,1)$ and $(3,3)$

(d) On the same set of axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, labelling all axis-intercepts and intersection points with their coordinates.

(4 marks)



End of questions

Question 3 **(5 marks)**

Recall from Year 10 that you can use *Polynomial Long Division* to divide a polynomial $P(x)$ by a divisor $D(x)$, resulting in a quotient $Q(x)$ and a remainder $R(x)$. You probably wrote this as

$$P(x) = D(x)Q(x) + R(x)$$

You may also recall the *Remainder Theorem*:

When a polynomial $P(x)$ is divided by $(x - a)$, the remainder is $P(a)$.

You may even recall The *Factor Theorem*:

For a polynomial $P(x)$, if $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.

(a) Prove the *Remainder Theorem.* (3 marks)

$P(x) = (x-a)Q(x) + R(x)$

when $x = a$:

$P(a) = (a-a)Q(a) + R(a)$

$\therefore P(a) = R(a)$

$\therefore$ when $x = a$, the remainder $R(a) = P(a)$.

Q.E.D.

(b) Prove the *Factor Theorem.* (2 marks)

$P(x) = (x-a)Q(x) + P(a)$ from the remainder theorem

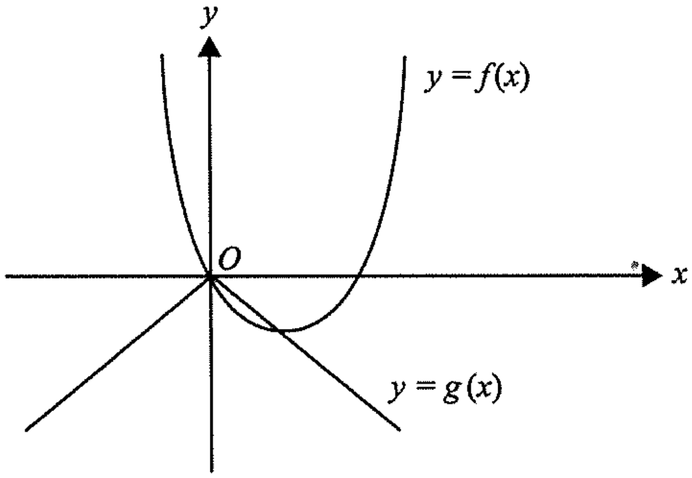
$\therefore P(x) = (x-a)Q(x)$ since $P(a) = 0$

$\therefore (x-a)$ is a factor of $P(x)$

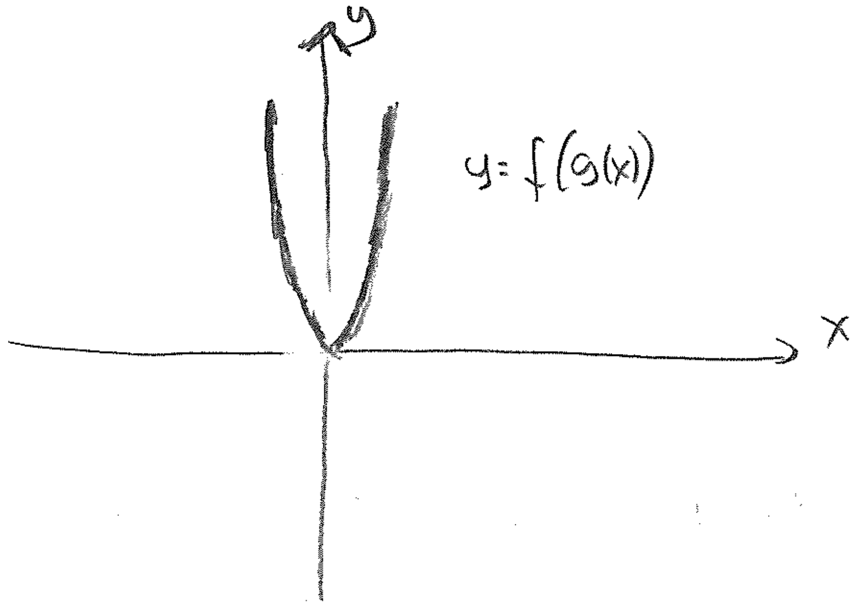
Q.E.D.

Question 4 **(5 marks)**

The graphs of $y = f(x)$ and $y = g(x)$ are as shown below.



Sketch the graph of $y = f(g(x))$.



$f(g(0)) = f(0) = 0$, so $(0,0)$ is on the graph

for $x \leq 0$: $g(x) \approx x$, so $f(g(x)) \approx f(x)$

for $x > 0$: $g(x) \approx -x$, so $f(g(x)) \approx f(-x)$

Question 5 **(7 marks)**

Let $f(x) = x^3 - x^2 + 3x - 3$.

(a) Show that $(x-1)$ is a factor of $f(x)$. (2 marks)

$f(1) = 1 - 1 + 3 - 3 = 0$

So, by the factor theorem, $x-1$ is a factor of $f(x)$.

(b) Hence, or otherwise, find all the solutions to the equation $f(x) = 0$. (3 marks)

$$\begin{array}{r|l} & x^2 + 3 \\ x-1 & x^3 - x^2 + 3x - 3 \\ & x^3 - x^2 \\ \hline & 0 + 3x \\ & 3x - 3 \\ \hline & 0 \end{array}$$

$x^2 + 3 = 0$

$\Downarrow$

$x = \pm\sqrt{3}\,i$

The three solutions are:

$x = 1, -\sqrt{3}i, \sqrt{3}i$

(c) In part (b) you would have noticed that two of the solutions (roots) were complex conjugates of each other. When will a cubic equation have two **non-real** solutions that are complex conjugates of each other? (2 marks)

When there is only one real solution $x = a$, cubic can be factorised as $(x-a)Q(x)$, whee $Q(x)$ is of degree two.

Since $Q(x)$ has no real solution, the two solutions obtained by the quadratics formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ will result in two non-real solutions that are complex conjugates of each other.

Question 6 **(8 marks)**

Let $f(x) = x - \frac{1}{2}x^2$ for $x \le a$.

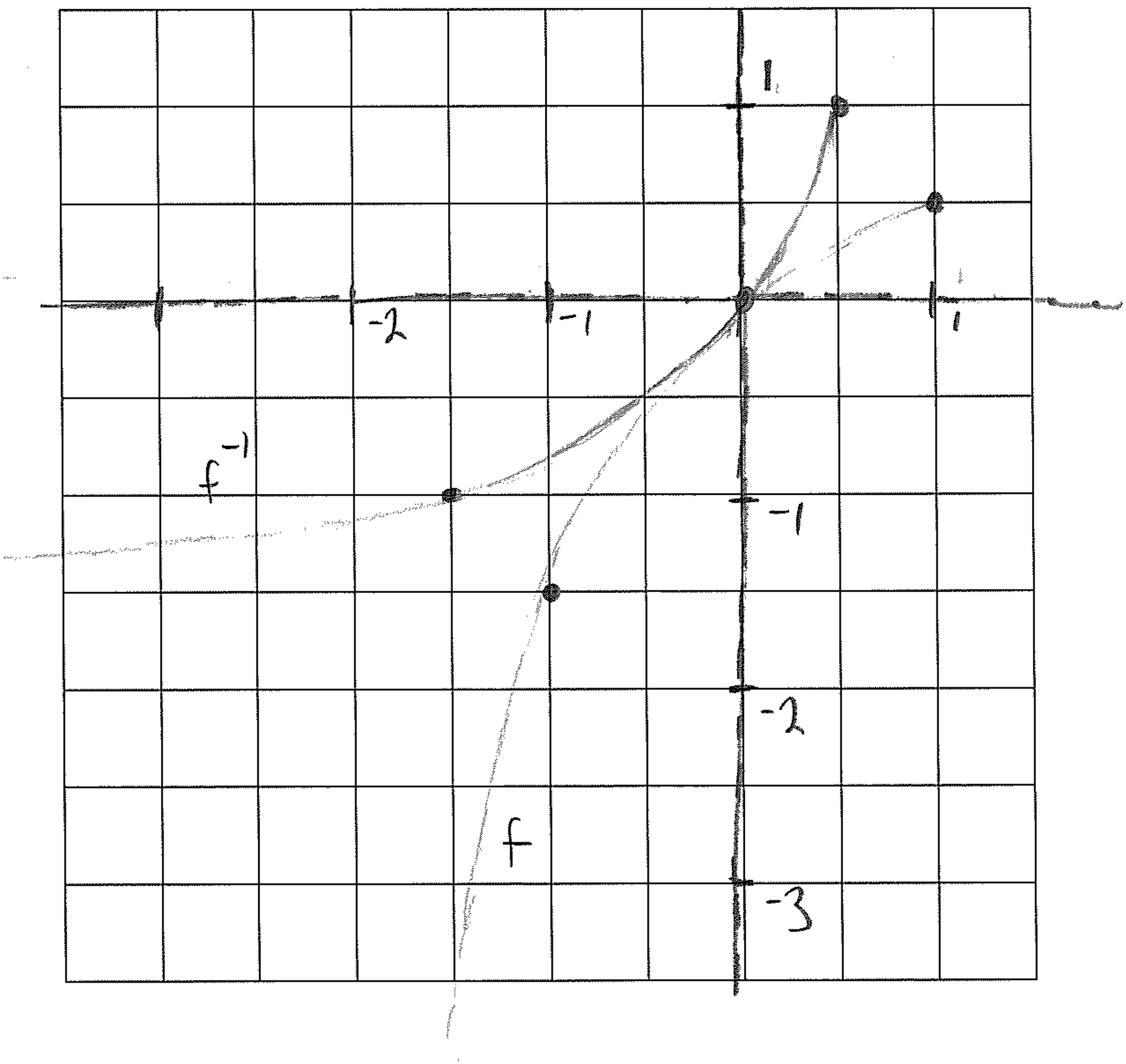
(a) Determine the largest value of a such that $f(x)$ has an inverse function. (2 marks)

$f(x) = \frac{1}{2}(2x - x^2)$

$= \frac{1}{2}x(2-x)$, i.e. turning point for $x = 1$,

$\therefore a = 1$

(b) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes. (Use the same scale on both axes) (3 marks)



(c) Find an expression for $f^{-1}(x)$. (3 marks)

$$y = x - \frac{1}{2}x^2$$

$$x = y - \frac{1}{2}y^2$$

$$\frac{1}{2}y^2 - y + x = 0$$

$$y^2 - 2y + 2x = 0$$

$$\therefore \quad y = \frac{2 \pm \sqrt{4 - 4 \times 1 \times (2x)}}{2}$$

$$= 1 \pm \sqrt{1-2x}$$

$$\therefore \quad f^{-1}(x) = 1 - \sqrt{1-2x}$$

(Because (0,0) is on the graph ...)

Question 7 **(5 marks)**

Consider the function

$$f(x) = \frac{x+1}{x-1}, x \neq 1.$$

(a) Show that $f(f(x)) = x$. (2 marks)

$$f(f(x)) = \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1}$$

$$= \frac{\frac{x+1}{x-1} + \frac{x-1}{x-1}}{\frac{x+1}{x-1} - \frac{x-1}{x-1}}$$

$$= \frac{2x}{2}$$

$$= x$$

(b) Use algebra to find an expression for the inverse function, $f^{-1}(x)$. (3 marks)

$$y = \frac{x+1}{x-1}$$

$$x = \frac{y+1}{y-1}$$

$$(y-1)x = y+1$$

$$xy - x = y+1$$

$$xy - y = x+1$$

$$y(x-1) = x+1$$

$$y = \frac{x+1}{x-1} \qquad \therefore f^{-1}(x) = \frac{x+1}{x-1}, \ x \neq 1$$

End of questions